On γ-vectors and the derivatives of the tangent and secant functions

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Abstract

In this paper we consider the γ-vectors of the type A and B Coxeter complexes as well as the γ-vectors of the type A and B associahedrons. By using the derivatives of the tangent and secant functions, we provide a description for these γ-vectors.

Keywords: γ-vectors; Tangent function; Secant function; Eulerian polynomials

1 Introduction

Let $\mathfrak{S}_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. The *hyperoctahedral group* B_n is the group of signed permutations of the set $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i, where $\pm[n] = \{\pm 1, \pm 2, \ldots, \pm n\}$. A permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathfrak{S}_n$ is *alternating* if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. Similarly, an element π of B_n is alternating if $\pi(1) > \pi(2) < \pi(3) > \cdots \pi(n)$. Denote by E_n and E_n^B the number of alternating elements in $\mathfrak{S}_n$ and B_n, respectively. It is well known (see [6, 25, 26]) that

$$\sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \tan x + \sec x,$$
$$\sum_{n=0}^{\infty} E_n^B \frac{x^n}{n!} = \tan 2x + \sec 2x.$$

A *descent* of a permutation $\pi \in \mathfrak{S}_n$ is a position i such that $\pi(i) > \pi(i+1)$, where $1 \leq i \leq n-1$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. Then the equations

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{\operatorname{des}(\pi)} = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k,$$

define the *Eulerian polynomials* $A_n(x)$ and the *Eulerian numbers* $\left\langle {n \atop k} \right\rangle$ (see [24, A008292]).

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For each $\pi \in B_n$, we define

$$\operatorname{des}_B(\pi) = \#\{i \in \{0, 1, 2, \ldots, n-1\} | \pi(i) > \pi(i+1)\},$$

where $\pi(0) = 0$. Let

$$B_n(x) = \sum_{\pi \in B_n} x^{\operatorname{des}_B(\pi)} = \sum_{k=0}^{n} B(n,k)x^k.$$

The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n,k)$ is called an *Eulerian number of type B* (see [24, A060187]).

Derivative polynomials are an important part of combinatorial trigonometry (see [8, 11, 13, 14, 15, 16, 17] for instance). Write $y = \tan(x)$ and $z = \sec(x)$. Denote by D the differential operator d/dx. Clearly, we have $D(y) = 1+y^2$ and $D(z) = yz$. In 1995, Hoffman [13] considered two sequences of *derivative polynomials* defined respectively by $D^n(y) = P_n(y)$ and $D^n(z) = zQ_n(y)$. From the chain rule it follows that the polynomials $P_n(u)$ satisfy $P_0(u) = u$ and $P_{n+1}(u) = (1+u^2)P_n'(u)$, and similarly $Q_0(u) = 1$ and $Q_{n+1}(u) = (1+u^2)Q_n'(u) + uQ_n(u)$. As shown in [13], their exponential generating functions

$$P(u,t) = \sum_{n=0}^{\infty} P_n(u)\frac{t^n}{n!} \quad \textit{and} \quad Q(u,t) = \sum_{n=0}^{\infty} Q_n(u)\frac{t^n}{n!}$$

are given by the explicit formulas

$$P(u,t) = \frac{u+\tan(t)}{1-u\tan(t)} \quad \textit{and} \quad Q(u,t) = \frac{\sec(t)}{1-u\tan(t)}. \tag{1}$$

Note that $D(y) = z^2$ and $D(z) = yz$. Assume that

$$(Dy)^{n+1}(y) = (Dy)(Dy)^n(y) = D(y(Dy)^n(y)),$$

$$(Dy)^{n+1}(z) = (Dy)(Dy)^n(z) = D(y(Dy)^n(z)).$$

Recently, we obtain the following result.

Theorem 1 ([17]). *For $n \geq 1$, we have*

$$(Dy)^n(y) = 2^n \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle y^{2n-2k-1}z^{2k+2},$$

$$(Dy)^n(z) = \sum_{k=0}^{n} B(n,k)y^{2n-2k}z^{2k+1}.$$

In the following discussion, we always write $f = \sec(2x)$ and $g = 2\tan(2x)$. In this paper we will consider the following differential system:

$$D(f) = fg, \quad D(g) = 4f^2. \tag{2}$$

Define $h = \tan(2x)$. Note that $f^2 = 1+h^2$ and $g = 2h$. So the following result is immediate.

Proposition 2. *For $n \geq 0$, we have*

$$D^n(f) = 2^n f Q_n(h), \quad D^n(g) = 2^{n+1} P_n(h).$$

The paper is organised as follows. In Section 2, we collect some notation, definitions and results that will be needed in the rest of the paper. In Section 3, we show that the γ-vectors of the type A and B Coxeter complexes can be respectively generated by $D^n(g)$ and $D^n(f)$. In Section 4, we show that the γ-vectors of the type A and B associahedrons can be respectively generated by $(fD)^n(g)$ and $(fD)^n(f)$.

2 Notation, definitions and preliminaries

Let W be a finite Coxeter group with simple reflections $s_1, s_2, \ldots, s_n$. Then a *descent* in some $\omega \in W$ may be defined as an index i such that $\ell(\omega s_i) < \ell(\omega)$, where $\ell(\omega)$ denotes the minimum length of an expression for ω as a product of simple reflections. The *Eulerian polynomial* $P(W, x)$ of a finite Coxeter group W is defined by

$$P(W, x) = \sum_{\omega \in W} x^{d(\omega)},$$

where $d(\omega)$ is defined to be the number of descents in ω. The polynomial $P(W, x)$ is also the h-polynomial of the Coxeter complex of W. In particular, for Coxeter groups of types A and B, we have $P(A_{n-1}, x) = A_n(x)$ and $P(B_n, x) = B_n(x)$.

Let Δ be a $(d-1)$-dimension simplicial complex. The *h-polynomial* of Δ is the generating function $h(\Delta; x) = \sum_{i=0}^{d} h_i(\Delta) x^i$ defined by the following identity:

$$\sum_{i=0}^{d} h_i(\Delta) x^i (1+x)^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta) x^i,$$

where $f_i(\Delta)$ is the number of faces of Δ of dimension i. If Δ is a simplicial homology sphere, then there exist integers γ_i such that

$$h(\Delta; x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1+x)^{d-2i}.$$

Following Gal [12], we call $(\gamma_0, \gamma_1, \ldots)$ the γ-vector of Δ. The *γ-polynomial* is defined by $\gamma(x) = \sum_{i \geq 0} \gamma_i x^i$. Various descriptions of γ-vectors have been pursued by several authors. The reader is referred to [1, 3, 7, 9, 20, 21, 22, 27] for recent progress on this subject.

Let $\pi = \pi(1)\pi(2) \cdots \pi(n) \in \mathfrak{S}_n$. An integer $i \in [n]$ is a *peak* if $\pi(i-1) < \pi(i) > \pi(i+1)$, a *double descent* if $\pi(i-1) > \pi(i) > \pi(i+1)$, where $\pi(0) = \pi(n+1) = 0$. Denote by $a(n, k)$ the number of permutations of $[n]$ with k peaks and without double descent. A type B slides of $\pi \in B_n$ is any decreasing run of $|\pi(1)| \cdots |\pi(n)|$ of length at least 2 (see [7]). Denote by $b(n, k)$ the number of elements of B_n with k descents and k slides.

Let us now recall two classical result.

Theorem 3 ([10, Théorème 5.6]). *For $n \geq 1$, we have*

$$A_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a(n,k)x^k(1+x)^{n-1-2k}.$$

Theorem 4 ([7, Theorem 4.7.]). *For $n \geq 1$, we have*

$$B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k)x^k(1+x)^{n-2k}.$$

As pointed out by Brändén [3] and Nevo and Petersen [20], it would be interesting to find various combinatorial descriptions of γ-vectors. In the next section, we provide a description for the γ-vectors of the type A and B Coxeter complexes.

3 γ-vectors generated by $D^n(f)$ and $D^n(g)$

It is well known that the numbers $a(n,k)$ satisfy the recurrence

$$a(n,k) = (k+1)a(n-1,k) + (2n-4k)a(n-1,k-1),$$

with the initial conditions $a(1,0) = 1$ and $a(1,k) = 0$ for $k \geq 1$ (see [24, A101280]), and the numbers $b(n,k)$ satisfy the recurrence

$$b(n,k) = (2k+1)b(n-1,k) + 4(n+1-2k)b(n-1,k-1), \tag{3}$$

with the initial conditions $b(1,0) = 1$ and $b(1,k) = 0$ for $k \geq 1$ (see [7, Section 4]).

Define the generating functions

$$a_n(x) = \sum_{k \geq 0} a(n,k)x^k,$$

$$b_n(x) = \sum_{k \geq 0} b(n,k)x^k.$$

The first few $a_n(x)$ and $b_n(x)$ are respectively given as follows:

$$a_1(x) = 1, a_2(x) = 1, a_3(x) = 1 + 2x, a_4(x) = 1 + 8x;$$

$$b_1(x) = 1, b_2(x) = 1 + 4x, b_3(x) = 1 + 20x, b_4(x) = 1 + 72x + 80x^2.$$

Combining (1) and [7, Prop. 3.5, Prop. 4.10], we immediately get the following result.

Theorem 5. *For $n \geq 1$, we have*

$$a_n(x) = \frac{1}{x}\left(\frac{\sqrt{4x-1}}{2}\right)^{n+1} P_n\left(\frac{1}{\sqrt{4x-1}}\right),$$
$$b_n(x) = (4x-1)^{\frac{n}{2}} Q_n\left(\frac{1}{\sqrt{4x-1}}\right).$$

We can now present the first main result of this paper.

Theorem 6. *For $n \geq 0$, we have*

$$D^n(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} b(n,k) f^{2k+1} g^{n-2k},$$

$$D^n(g) = 2^{n+1} \sum_{k=0}^{\lfloor n-1/2 \rfloor} a(n,k) f^{2k+2} g^{n-1-2k}.$$

Proof. We only prove the assertion for $D^n(f)$ and the corresponding assertion for $D^n(g)$ follows from similar consideration. It follows from (2) that $D(f) = fg$ and $D^2(f) = fg^2 + 4f^3$. For $n \geq 0$, we define $\widetilde{b}(n,k)$ by

$$D^n(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{b}(n,k) f^{2k+1} g^{n-2k}, \tag{4}$$

Then $\widetilde{b}(1,0) = 1$ and $\widetilde{b}(1,k) = 0$ for $k \geq 1$. It follows from (4) that

$$D(D^n(f)) = \sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1)\widetilde{b}(n,k) f^{2k+1} g^{n-2k+1} + 4 \sum_{k=0}^{\lfloor n/2 \rfloor} (n-2k)\widetilde{b}(n,k) f^{2k+3} g^{n-2k-1}.$$

We therefore conclude that $\widetilde{b}(n+1,k) = (2k+1)\widetilde{b}(n,k) + 4(n+2-2k)\widetilde{b}(n,k-1)$ and complete the proof by comparing it with (3). $\square$

4 γ-vectors generated by $(fD)^n(f)$ and $(fD)^n(g)$

In this section, we present a description of the γ-vectors of the type A and B associahedrons.

The well known h-polynomials of the type A and B associahedrons are respectively given as follows (see [2, 19, 22, 23] for instance):

$$h(\Delta_{FZ}(A_{n-1}), x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} x^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} C_k \binom{n-1}{2k} x^k (1+x)^{n-1-2k},$$

$$h(\Delta_{FZ}(B_n), x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{2k}{k} \binom{n}{2k} x^k (1+x)^{n-2k},$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the kth *Catalan number* and the coefficients of x^k of $h(\Delta_{FZ}(A_{n-1}), x)$ is the *Narayana number* $N(n, k+1)$.

Define

$$F(n,k) = C_k \binom{n-1}{2k}, \quad H(n,k) = \binom{2k}{k} \binom{n}{2k}.$$

There are many combinatorial interpretations of the number $F(n,k)$, such as $F(n,k)$ is number of *Motzkin paths* of length $n-1$ with k up steps (see [24, A055151]). It is easy to verify that the numbers $F(n,k)$ satisfy the recurrence relation

$$(n+1)F(n,k) = (n+2k+1)F(n-1,k) + 4(n-2k)F(n-1,k-1),$$

with initial conditions $F(1,0) = 1$ and $F(1,k) = 0$ for $k \geq 1$, and the numbers $H(n,k)$ satisfy the recurrence relation

$$nH(n,k) = (n+2k)H(n-1,k) + 4(n-2k+1)H(n-1,k-1), \tag{5}$$

with initial conditions $H(1,0) = 1$ and $H(1,k) = 0$ for $k \geq 1$ (see [24, A089627]).

Assume that

$$(fD)^{n+1}(f) = (fD)(fD)^n(f) = fD((fD)^n(f)),$$
$$(fD)^{n+1}(g) = (fD)(fD)^n(g) = fD((fD)^n(g)).$$

Now we present the second main result of this paper.

Theorem 7. *For $n \geq 1$, we have*

$$(fD)^n(f) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} H(n,k) f^{n+1+2k} g^{n-2k},$$
$$(fD)^n(g) = 2(n+1)! \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} F(n,k) f^{n+2+2k} g^{n-1-2k}.$$

Proof. We only prove the assertion for $(fD)^n(f)$ and the corresponding assertion for $(fD)^n(g)$ follows from similar consideration. It follows from (2) that $(fD)(f) = f^2 g$ and $(fD)^2(f) = 2(f^3g^2 + 2f^5)$. For $n \geq 0$, we define $\widetilde{H}(n,k)$ by

$$(fD)^n(f) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{H}(n,k) f^{n+2k+1} g^{n-2k}, \tag{6}$$

Then $\widetilde{H}(1,0) = 1$ and $\widetilde{H}(1,k) = 0$ for $k \geq 1$. It follows from (6) that

$$\frac{(fD)^{n+1}(f)}{n!} = \sum_{k=0}^{\lfloor n/2 \rfloor} \widetilde{H}(n,k)(n+2k+1) f^{n+2k+2} g^{n-2k+1} + \sum_{k=0}^{\lfloor n/2 \rfloor} 4\widetilde{H}(n,k)(n-2k) f^{n+2k+4} g^{n-2k-1}.$$

We therefore conclude that

$$(n+1)\widetilde{H}(n+1,k) = (n+2k+1)\widetilde{H}(n,k) + 4(n-2k+2)\widetilde{H}(n,k-1)$$

and complete the proof by comparing it with (5). □

Define

$$N_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} \binom{n}{k} \binom{n}{k+1} (x+1)^k (x-1)^{n-1-k},$$
$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 (x+1)^k (x-1)^{n-k}.$$

It should be noted that the polynomial $\frac{1}{2^n} L_n(x)$ is the Legendre polynomial [24, A100258]. Taking $f^2 = 1 + h^2$ and $g = 2h$ in Theorem 7 leads to the following result and we omit the proof of it, since it is a straightforward verification.

Corollary 8. *For $n \geq 1$, we have*

$$(fD)^n(f) = n!f^{n+1}(-\imath)^n L_n(\imath h),$$
$$(fD)^n(g) = 2(n+1)!f^{n+2}(-\imath)^{n-1} N_n(\imath h),$$

where $\imath = \sqrt{-1}$.

5 Concluding remarks

Many combinatorial objects permit a description using the notion of context-free grammars (see [4, 5, 18] for instance). The grammatical method was introduced by Chen [4] in the study of exponential structures in combinatorics. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. A *context-free grammar* G over A is defined as a set of substitution rules that replace a letter in A by a formal function over A. The formal derivative D is a linear operator defined with respect to a context-free grammar G. Hence Theorem 6 and Theorem 7 can be respectively restated by using the grammar $G_1 = \{x \rightarrow xy, y \rightarrow 4x^2\}$ and the grammar $G_2 = \{x \rightarrow x^2y, y \rightarrow 4x^3\}$.

We end this paper by giving another description of the γ-vectors of the type A Coxeter complex.

Theorem 9. *If $G = \{x \rightarrow xy, y \rightarrow 2x\}$, then*

$$D^n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} a(n+1,k)x^{k+1}y^{n-2k}.$$

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